SOLUTION EXERCISE SHEET 5

Exercise 1. We begin by observing the following formulas for the functions sin, cos, cosh and sinh, which follow directly through the definition. For every $z \in \mathbb{C}$ we have that

$$\begin{aligned}\cos(z) &= \frac{e^{iz}+e^{-iz}}{2},\\ \sin(z) &= \frac{e^{iz}-e^{-iz}}{2i},\\ \cosh(z) &= \frac{e^{z}+e^{-z}}{2},\\ \sinh(z) &= \frac{e^{z}-e^{-z}}{2}.\end{aligned}$$

Then we have the following simple computations, which prove the formulas (a)-(g).

(a) Let $z, w \in \mathbb{C}$ we have

$$\sin(z)\cos(w)+\cos(z)\sin(w) = \frac{e^{iz}-e^{-iz}}{2i}\frac{e^{iw}+e^{-iw}}{2}+\frac{e^{iz}+e^{-iz}}{2}\frac{e^{iw}-e^{-iw}}{2i}$$
$$= \frac{e^{i(z+w)}-e^{-i(z+w)}+e^{iz-iw}-e^{-iz+iw}}{4i}+\frac{e^{i(z+w)}-e^{-i(z+w)}-e^{iz-iw}+e^{-iz+iw}}{4i}$$
$$= \frac{e^{i(z+w)}-e^{-i(z+w)}}{2i} = \sin(z+w).$$

(b) Let $z, w \in \mathbb{C}$ we have

$$\cos(z)\cos(w)-\sin(z)\sin(w) = \frac{e^{iz}+e^{-iz}}{2}\frac{e^{iw}+e^{-iw}}{2}-\frac{e^{iz}-e^{-iz}}{2i}\frac{e^{iw}-e^{-iw}}{2i}$$
$$= \frac{e^{i(z+w)}+e^{-i(z+w)}+e^{iz-iw}+e^{-iz+iw}}{4}+\frac{e^{i(z+w)}+e^{-i(z+w)}-e^{iz-iw}-e^{-iz+iw}}{4}$$
$$= \frac{e^{i(z+w)}+e^{-i(z+w)}}{2} = \cos(z+w).$$

(c) The prove is almost the same than in (a).

(d) The prove is almost the same than in (b).

(e) Let $z \in \mathbb{C}$ we have

$$\cos(z)+i\sin(z) = \frac{e^{iz}+e^{-iz}}{2}+i\frac{e^{iz}-e^{-iz}}{2i} = \exp(iz).$$

In case of $z \in \mathbb{R}$ this is known as Euler's formula.

(f) Let $z \in \mathbb{C}$ we have

$$\begin{aligned}\sin(z)^2+\cos(z)^2 &= \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2+\left(\frac{e^{iz}+e^{-iz}}{2}\right)^2\\ &= \frac{e^{i2z}-2+e^{-i2z}}{-4}+\frac{e^{i2z}+2+e^{-i2z}}{4}=1.\end{aligned}$$

(g) Let $z \in \mathbb{C}$ we have

$$\begin{aligned}\cosh(z)^2-\sinh(z)^2 &= \left(\frac{e^{z}+e^{-z}}{2}\right)^2-\left(\frac{e^{z}-e^{-z}}{2}\right)^2\\ &= \frac{e^{2z}+2+e^{-2z}}{4}-\frac{e^{2z}-2+e^{-2z}}{4}=1.\end{aligned}$$

Exercise 2. In this exercise we need to use properties of the n-th roots of unity. First we observe the following in the simple situations of $k = 1, 2, 3, 4$.

(a) If $k = 1$, then by definition for any $z \in \mathbb{C}$

$$\sum_{n=0}^{\infty}\frac{z^n}{n!}=\exp(z).$$

(b) If $k = 2$ we get by evaluating the exponential in z and $-z$ for some fixed $z \in \mathbb{C}$ that

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{z^n}{n!}+\sum_{n=0}^{\infty}\frac{(-z)^n}{n!}=2\sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!} \iff\\ &\sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!}=\frac{\exp(z)+\exp(-z)}{2}=\cosh(z).\end{aligned}$$

(c) If $k = 3$, then we try to generalize that we did above. Observe that 1 and -1 are the two 2 roots of unity. Let $\xi_1 := \exp(\frac{2\pi i}{3})$, which is the 3 roots of unity with the smallest positive angle. The two other ones are $\xi^0 = 1$ and $\xi_2 = \xi_1^2$. Observe that $1+\xi_1^m+\xi_2^m = 0$ for $m = 1, 2$. We get for any $z \in \mathbb{C}$ fixed, by evaluating the exponential in z, $\xi_1 \cdot z$ and $\xi_2 \cdot z$ that

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{z^n}{n!}+\sum_{n=0}^{\infty}\frac{(\xi_1\cdot z)^n}{n!}+\sum_{n=0}^{\infty}\frac{(\xi_2\cdot z)^n}{n!}\\ &=3\sum_{n\in3\mathbb{N}}\frac{z^n}{n!}+(1+\xi_1+\xi_2)\sum_{n\in3\mathbb{N}+1}\frac{z^n}{n!}+(1+\xi_1^2+\xi_2^2)\sum_{n\in3\mathbb{N}+2}\frac{z^n}{n!}\\ &=3\sum_{n=0}^{\infty}\frac{z^{3n}}{(3n)!}\iff\sum_{n=0}^{\infty}\frac{z^{3n}}{(3n)!}=\frac{\exp(z)+\exp(\xi_1\cdot z)+\exp(\xi_2\cdot z)}{3}.\end{aligned}$$

(d) If $k = 4$, then the situation is very similar to $k = 3$. The four 4 roots of unity are $1, \xi_1 = i, \xi_2 = -1, \xi_3 = -i$. We have again that $1+\xi_1^m+\xi_2^m+\xi_3^m = 0$ for

$m = 1, 2, 3$ and we get for $z \in \mathbb{C}$ fixed that

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} + \sum_{n=0}^{\infty} \frac{(\xi_1 \cdot z)^n}{n!} + \sum_{n=0}^{\infty} \frac{(\xi_2 \cdot z)^n}{n!} + \sum_{n=0}^{\infty} \frac{(\xi_3 \cdot z)^n}{n!}$$
$$= \sum_{m=0}^{3} \left(\sum_{j=0}^{3} \xi_j^m \cdot \sum_{n \in 4\mathbb{N}+m} \frac{z^n}{n!} \right)$$
$$= 4 \sum_{n \in 4\mathbb{N}} \frac{z^n}{n!} \iff \sum_{n=0}^{\infty} \frac{z^{4n}}{(4n)!} = \frac{\exp(z) + \exp(i \cdot z) + \exp(-z) + \exp(-iz)}{4}$$
$$= \sinh(z)\cos(z),$$

where $\xi_0 = 1$.

In order to derive a general formula we observe that if $\xi \in \mathbb{C} \setminus \{1\}$ is a non trivial n-th root of unity, then we have that $\sum_{m=0}^{n-1} \xi^m = \frac{1-\xi^n}{1-\xi} = 0$. Further, if $\xi := \exp(\frac{2\pi i}{n})$, then all roots are given by $\xi_m = \exp(\frac{2\pi m i}{n}) = \xi^m$ for $m = 0, \ldots, n-1$. This shows that $\xi_m^\ell = \xi^{m \cdot \ell}$ for any $m, \ell = 0, \ldots, n-1$. Therefor, for $k \in \mathbb{N}^*$ and $z \in \mathbb{C}$ we get that

$$\sum_{j=0}^{k-1} \sum_{n=0}^{\infty} \frac{(\xi_j \cdot z)^n}{n!} = \sum_{m=0}^{k-1} \left(\sum_{j=0}^{k-1} \xi_j^m \cdot \sum_{n \in k\mathbb{N}+m} \frac{z^n}{n!} \right)$$
$$\sum_{m=0}^{k-1} \left(\sum_{j=0}^{k-1} \xi_m^j \cdot \sum_{n \in k\mathbb{N}+m} \frac{z^n}{n!} \right) = k \sum_{n \in k\mathbb{N}} \frac{z^n}{n!} \iff$$
$$\sum_{n=0}^{\infty} \frac{z^{k \cdot n}}{(k \cdot n)!} = \frac{1}{k} \sum_{m=0}^{k-1} \exp(\xi_m \cdot z).$$

Exercise 3. Clearly the radius of convergence is 1 as we have that $\sum_{n=0}^{\infty} |z|^{n!} \leq \sum_{n=0}^{\infty} |z|^n < \infty$ if $|z| < 1$ and the sum equals $+\infty$ for $z = 1$. Next we show that if $\alpha \in \mathbb{Q}$, then $|f(re^{2\pi i \alpha})| \to +\infty$ when $r \to 1$, where $f(z) := \sum_{n=0}^{\infty} z^{n!}$. This is stronger than just being unbounded.

In order to prove that we observe that as $\alpha \in \mathbb{Q}$ there exits p, q relatively prime such that $\alpha = \frac{p}{q}$ and that for $n \geq q$ we have that $q | n!$. Thus we have that $\alpha \cdot n! \in \mathbb{Z}$ for any $n \geq q$. We get for $r \in (0, 1)$ that

$$f(re^{2\pi i \alpha}) = \sum_{n=0}^{\infty} r^{n!} e^{2\pi i n! \alpha} = \sum_{n=0}^{q-1} r^{n!} e^{2\pi i n! \alpha} + \sum_{n=q}^{\infty} r^{n!}.$$

Next we observe that there exist $M > 0$ such that $|\sum_{n=0}^{q-1} r^{n!} e^{2\pi i n! \alpha}| \leq M$ for all $r \in [0, 1]$ and therefore

$$|f(re^{2\pi i \alpha})| \geq \left| \sum_{n=q}^{\infty} r^{n!} \right| - \left| \sum_{n=0}^{q-1} r^{n!} e^{2\pi i n! \alpha} \right| \geq \sum_{n=q}^{\infty} r^{n!} - M \xrightarrow{r \to 1} \infty.$$

This proves the claim.